M.Sc. (Mathematics) (New CBCS Pattern) Semester-I

PSCMTH02 - Real Analysis

P. Pages : 2 **GUG/S/25/13738**

Time : Three Hours * 2 2 1 3 * Max. Marks : 100

Notes : 1. Solve all **five** questions.
2. Each question carry equal marks.

UNIT – I

1. a) Prove that the sequence of functions $\{f_n\}$, define on E, converges uniformly on E if and only if for every $\in > 0$ there exists an integer N such that $m \ge N, n \ge N,\ x \in E$ implies $|f_n(x) - f_m(x)| \le \in$. **10**

b) State and prove the Stone- Weierstrass Theorem. **10**

OR

c) Suppose $\{f_n\}$ is a sequence of functions, differentiable on [a, b] and such that $\{f_n(x_0)\}$ converges for some point x_0 on [a, b]. If $\{f_n^{'}\}$ converges uniformly on [a, b], then prove that $\{f_n\}$ converges uniformly on $[a, b]$, to a function f, and f' $f'(x) = \lim\limits_{n\to\infty} f_n^{'}(x)$, $(a \le x \le b)$. **10**

d) Let α be monotonically increasing on $[a, b]$. Suppose $f_n \in \Re(\alpha)$ on [a, b], for n = 1,2,3,…, and suppose $f_n \to f$ uniformly on [a, b]. Then prove that $f \in \Re(\alpha)$ on [a, b], and $\int_a^b f\, d\alpha = \lim\limits_{n\to\infty} \int_a^b f_n d\alpha$. **10**

UNIT – II

2. a) If X is a complete metric space, and if ϕ is a contraction of X into X, then prove that there exists one and only one $x \in X$ such that $\phi(x) = x$. **10**

b) State and prove the inverse function theorem. **10**

OR

c) Suppose f maps an open set $E \subset R^n$ into R^m. The prove that $f \in \wp(E)$ if and only if the partial derivatives $D_j f_i$ exist and are continuous on E for $1 \le i \le m, 1 \le j \le n$. **10**

d) Suppose E is an open set in R^n, f maps E into R^m, f is differentiable at $x_0 \in E, g$ maps an open set containing f(E) into R^k, and g is differentiable at $f(x_0)$. The prove that the mapping F of E into R^k.
defined by $F(x) = g(f(x))$ is differentiable at x_0, and F' $F'(x_0) = g'(f(x_0)) f'(x_0)$. **10**

UNIT – III

3. a) Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas on a locally Euclidean space. If two charts (V, Ψ) and (W, σ) are both compatible with the atlas $\{(U_\alpha, \phi_\alpha)\}$, then prove that they are compatible with each other. **10**

b) Prove that any atlas $u = \{(U_\alpha, \phi_\alpha)\}$ on a locally Euclidean space is contained in a unique maximal atlas. **10**

OR

c) For any two positive integers m and n, let $\mathbb{R}^{mxn}$ be the vector space of all $m \times n$ matrices. Since $\mathbb{R}^{mxn}$ is isomorphic to $\mathbb{R}^{mn}$, we give it's the topology of $\mathbb{R}^{mn}$. Then prove that the general linear groups GL $(n, \mathbb{R})$ and $GL(n, \mathbb{C})$ are manifolds. **10**

d) Define **10**
(i) Locally Euclidean space of dimension n (ii) Coordinate neighbourhood
(iii) $\mathbb{C}^\infty$ - compatible charts.

UNIT – IV

4. a) State and prove that inverse function theorem for manifolds. **10**

b) If (U, ϕ) is a chart on a manifold M of dimension n, then prove that the coordinate map $\phi : U \to \phi(U) \subset \mathbb{R}^n$ is a diffeomorphism. **10**

OR

c) Suppose $F : N \to M$ is $\mathbb{C}^\infty$ at $p \in N$. If (U, ϕ) is any chart about p in N and (V, Ψ) is any chart about F(p) in M, then prove that $\Psi \cdot F \cdot \phi^{-1}$ is $\mathbb{C}^\infty$ at $\phi(p)$. **10**

d) Define Lie group and show that $GL(n, \mathbb{R})$ is a Lie group. **10**

5. a) Define pointwise convergence and uniform convergence for sequence of functions. **5**

b) Define differentiable function for functions of several variables. **5**

c) Define (i) topological manifold (ii) smooth manifold. **5**

d) Define (i) diffeomorphism (ii) smooth functions on a manifold. **5**

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